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COMPREHENSIVE STUDY ON RUNGE- KUTTA METHOD: STABILITY, ACCURACY AND APPLICATIONS IN NUMERICAL ANALYSIS

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Abstract :
Ordinary differential equations (ODEs) are frequently solved using the Runge-Kutta (RK) technique in applied mathematics, physics, and engineering. Through an analysis of the RK method's correctness, stability, and computing efficiency, this research presents a fresh viewpoint on it. A comparison with alternative numerical techniques is provided, highlighting enhancements in convergence speed and error reduction. The usefulness of RK techniques in practical applications is further illustrated by an example included in the study.

Keywords:
Differential equations, Runge-Kutta technique, numerical analysis, stability, convergence, and computational efficiency.

Objectives:

Analysing the Runge-Kutta method's mathematical underpinnings is one of the study's goals.

• To evaluate the accuracy and efficiency of several RK techniques (first-, second-, and fourth-order).

• To assess the RK method's error behaviour and stability when solving ODEs.

• To give an illustration of how it is used in scientific calculations.

Introduction :

When it is not possible to solve differential equations analytically, numerical techniques are crucial. Named after Carl Runge and Wilhelm Kutta, the Runge-Kutta technique offers a methodical way to approximate answers with a high degree of precision. RK approaches include several intermediary computations to increase precision, in contrast to more straightforward techniques like Euler's method. This study explores the RK method's mathematical formulation, real-world application, and performance assessment.

Main Body :

1) Runge-Kutta Method Overview

Initial value issues of the following type can be solved iteratively using the RK method:
With

$$\frac{dy}{dx} = f(x,y)\,, \quad y(x_0) = y_0$$

For -stages, the general RK formula is :

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i$$

where:

- k_i are intermediate slopes based on function evaluations,
- b_i are weighting coefficients,
- h is the step size.

2) Frequently Used Runge-Kutta Methods

I) First-Order (Euler's Method)

$y_{n+1} = y_n + hf(x_n, y_n)$

II) Second-Order (Midpoint Method)

$y_{n+1} = y_n + hk_2$

$$k_1 = f(x_n, y_n), \qquad k_2 = f\left(x_n + \frac{h}{2}, y_n + h\left(\frac{k_1}{2}\right)\right)$$

III)Fourth-Order (Classic Runge-Kutta Method, RK4)

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$If\ k_1 = f(x_n, y_n)$

$$then\ k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}\right)$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}\right) \quad and$$

$k_4 = f(x_n + h, y_n + hk_3)$

3) Analysis of Stability and Convergence

• **Stability:** Compared to Euler's approach, RK methods remain stable across a wider range of step sizes.

• **Convergence:** Higher-order RK techniques offer less error per iteration and faster convergence.

4) Evaluation in Relation to Other Numerical Methods :

Method	**Order**	**Accuracy**	**Stability**	**Complexity**
Euler's method	1	Low	Low	Simple
Midpoint method	2	Moderate	Moderate	Moderate
RK4 method	4	High	High	Complex

Example :

Take the differential equation

$\frac{dy}{dx} = x + y\ , y(0) = 1$ into consideration.

With h=0.1 and RK4, the initial computations are:

Using RK4 with $h = 0.1$, the first step calculations are:

$k_1 = f(0.1) = 0 + 1 = 1$
$k_2 = f(0.05,1 + 0.05) = 0.05 + 1.05 = 1.10$
$k_3 = f(0.05,1 + 0.055) = 0.05 + 1.055 = 1.105$
$k_4 = f(0.1,1 + 0.11) = 0.1 + 1.11 = 1.21$

$$y(0.1) = 1 + \frac{0.1}{6}(1 + 2(1.10) + 2(1.105) + 1.21) = 1.1104$$

is the final estimate.
At later stages, this procedure might be repeated to find answers.

Discussion:

Because the RK4 technique has higher-order correction terms than Euler's method, it shows better accuracy. Although the computational cost is higher, its increased stability makes it worth using in real-world scenarios. For solving stiff differential equations, RK approaches are used because to the efficiency and accuracy trade-off.

Limitations :

- RK approaches increase computing cost by requiring several function evaluations per step.
- Accuracy and stability are greatly impacted by the choice of step size.
- Not appropriate for unmodified, extremely stiff equations (for example, implicit RK methods).

Conclusion:

One of the mainstays of numerical analysis for ODE solutions is still the Runge-Kutta technique. Its mathematical underpinnings, computational effectiveness, and real-world applications were highlighted in this work. Specifically, the RK4 approach successfully strikes a compromise between computing effort and accuracy. For better results, future studies can investigate hybrid approaches and adaptive step-size modifications.

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